Force correlations and arches formation in granular assemblies

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(February 1, 2008)

In the context of a simple microscopic schematic scalar model we study the effects of spatial correlations in force transmission in granular assemblies. We show that the parameters of the normalized weights distribution function, $P(v) \sim v^{\alpha} \exp(-v/\phi)$, strongly depend on the spatial extensions, ξ_V, of such correlations. In particular we evaluate the functions $\phi(\xi_V)$ and $\alpha(\xi_V)$. If $\xi_V \to \infty$ then $\phi \sim \xi_V$ and weights are power law distributed.

Photoelastic visualizations of stress propagation in granular media have provided evidence of strong inhomogeneities in the distribution of contact forces with the formation of "force chains" which typically extend on space scales much larger than the dimensions of single grains [1–3]. These inhomogeneities results from local fluctuations and are accompanied by broad stress distributions observed in bead packs [2–4] which are responsible for many unusual properties of granular media [4].

Several theoretical approaches have been proposed to describe the unusual properties of stress patterns in such systems [4–7] and recently a schematic scalar model, which takes into account just the vertical component of the stress tensor, has been introduced to describe the statistics of stress distribution [2,8]. The mean field study of this model has shown rich results in good correspondence with experimental observations. However recent experiments [3] show the necessity to go beyond the mean field approximation and to take into account spatial correlations present in force transmission from grain to grain, correlations which actually lead to chains formation.

In the present paper, we study the effects of such correlations in the framework of a schematic scalar model, as in ref. [2,8], in which grains, disposed on a lattice, discharge their weights in random fractions, due to the disorder present in a typical granular assembly, on their bottom neighbours. The disorder of the environment, however, does not imply that force transmission between grains is uncorrelated [5,9,12]. In the complex dynamical process which leads an assembly to a static configuration, a certain fraction of sites carrying a strong shear (typically involved in the formation of "force chains"), slips away from some of their own neighbours, losing contact with them. These processes may induce strong spatial correlations in force transmission, which we try to take into account here.

In our model the weight, $w(i,h)$, supported by a grain at height h and column i of a square lattice (see Fig. 1) is (using the notation of ref. [8,9]):

$$\begin{aligned} w(i,h) = {} & q_+(i-1,h-1)\ w(i-1,h-1) \\ & +q_-(i+1,h-1)\ w(i+1,h-1) + 1 \end{aligned} \tag{1}$$

where q_+ (resp. q_-) is the fraction of the weight that site $(i-1,h-1)$ (resp. $(i+1,h-1)$) discharges on site (i,h), and we have supposed that the masses of single grains are equal to unity. Due to the conservation of the mass, we have the constraint $q_+(i,h)+q_-(i,h)=1$. The values of $q_+(i,h)$ are generally uniformly distributed in the interval $[0,1]$. However, we suppose that a fraction $1-\delta$ ($\delta \in [0,1]$) of the total number of grains (a fraction of grains randomly displaced on the lattice) is subjected to the slip condition proposed in ref. [9]:

$$q_+(i,h) = \theta(x) \tag{2}$$

Here θ is the Heaviside step function and x is [10]:

$$\begin{aligned} x = {} & [q_+(i-1,h-1)\ w(i-1,h-1) \\ & -q_-(i+1,h-1)\ w(i+1,h-1)]\,/w(i,h) \end{aligned} \tag{3}$$

Eq. (2) schematically expresses the slip condition in which, due to friction, a grain, strongly "pressed" from its top right (resp. left) neighbour, may discharge its weight mainly on its own right (resp. left) lower neighbour [11]. This mechanism originates force chains in our model.

Our tuning parameter is the fraction of grains not subject to the slip condition, δ, a parameter which, in real samples, is related to friction coefficients, elastic constants, grains shapes. The model interpolates the two extreme situations corresponding to a complete uncorrelated weight transmission, studied in ref. [8], which is obtained when $\delta = 1$, and the strongly correlated case studied in ref. [9], which is recovered in the limit $\delta = 0$.

This two dimensional model is simple enough to visualize mechanisms underlying force transmission. Actually, the sites which undergo the above "slip condition" are typical points enhancing correlations in force transmission and so belonging to "force" chains (see Fig. 1). In our case the "slipping" grains are randomly distributed on the lattice; thus the probability to have a segment of such grains of total length d, i.e., to have a chain of length d, is approximately $\mathcal{P}(d) \sim (1-\delta)^d$. Consequently the average chain length, ξ_V, which is experimentally easier to measure, is related to the fraction of such sites as: $\xi_V \simeq (1-\delta)/\delta$. The kind of correlation we have introduced above essentially generates "chains" which sustain the weight of the lattice columns which intersect them from above, as "arches" [5,7]. Thus, in the deep bulk of the system at depth L, the weight, W, that a point at the base of a chain of length d experiences, is approximately $W \sim Ld$. A quantity of theoretical as well as experimental and practical importance is the distribution $P(v)$ of normalized weights, $v = W/L$, felt by grains at a given depth L. With the above arguments we can derive an approximate expression to relate $P(v)$ to the fraction δ, i.e., to the average chain length ξ_V:

$$P(v) \equiv P(W(d)/L) = \mathcal{P}(d) \sim \exp(-v/\phi) \tag{4}$$

where $\phi \sim 1/\log[1/(1-\delta)]$ ($\phi \sim \xi_V$, if $\xi_V \to \infty$). This result implies important practical consequences, because it predicts that we have to expect to measure weight fluctuations of the order of ξ_V. Thus if ξ_V, which is a characteristic length of the specific system we consider, is very high, huge stress fluctuations arise in the sample. The above picture compares well with the numerical calculations we present below, where its consequences are discussed in more details.

Our numerical analysis of the present model concerns a square lattice of length $L = 10000$ and depth $M = 1000$, averaged over 10 different realizations. In this lattice we adopted periodic boundary conditions along the horizontal axis.

To study the effects of spatial correlations introduced by the slip condition of eq. (2), we evaluate the vertical space correlation function of forces, $C_V(r)$, along a main axis of our square lattice:

$$C_V(r) = \frac{\langle w(i,M)w(i-r,M-r)\rangle - w_m(0)w_m(r)}{\langle w(i,M)^2\rangle - w_m(0)^2} \tag{5}$$

with $w_m(r) = \langle w(i, M-r)\rangle$. The behaviour of $C_V(r)$, in the present model, is depicted in Fig. 2 as a function of the distance between grains r (expressed in lattice units). After a first jump from $C_V(0) = 1$ to $C_V(1) \sim 0.5$, $C_V(r)$ smoothly decreases approaching zero. The first part of the decay is exponential in r, and may be fitted with the following function:

$$C_V(r) = K_V \exp(-r/\xi_V) \tag{6}$$

The characteristic length ξ_V of eq. (6), is related to the measure of the extensions of forces inhomogeneities along the vertical direction in our system. It is the typical vertical extension of forces correlation, or, more crudely, the average vertical distance between crossing points of "stress paths" observed in photoelastic measurements (see Fig. 1). In Fig. 3, ξ_V is plotted as a function of δ [14]. In agreement with the above theoretical arguments, the approximate behaviour $\xi_V \sim \delta^{-1.0}$ is found, showing that ξ_V increases orders of magnitude when the fraction of sites, $1-\delta$, undergoing the slip condition approaches 1. In the limit $\xi_V \to \infty$, force chains extends over the whole system and this fact strongly affects weights distribution.

The horizontal space correlation function, $C_H(r)$, is defined as the average over the lower 10% of the system (to have more precise data) of $c(r,h)$:

$$c(r,h) = \frac{\langle w(i,h)w(i+r,h)\rangle - \langle w(i,h)\rangle^2}{\langle w(i,h)^2\rangle - \langle w(i,h)\rangle^2} \quad . \tag{7}$$

The function $C_H(r)$, depicted in Fig. 2, becomes negative as soon as $r \geq 1$, signaling that horizontally aligned grains are always slightly anti-correlated. When $r \to \infty$, $C_H(r)$ asymptotically approaches zero from below. As above, the first part of the decay is almost exponential:

$$C_H(r) = -K_H \exp(-r/\xi_H) \tag{8}$$

In this case, the length ξ_H corresponds to the typical horizontal spacing of chains. The function $\xi_H(\xi_V)$ is reported in Fig. 3. ξ_H diverges when $\delta \to 0$, and we approximately find $\xi_H \sim \xi_V^a$ with $a \sim 1/2$.

We now try to relate the above observations to the distribution, $P(v)$, of the weights, $v = w/w_m$, normalized by the mean $w_m = h$, supported by grains at a given depth h. The function $P(v)$, evaluated at the bottom layers in our model, is plotted in Fig. 4. In agreement with the above theoretical considerations, this quantity becomes independent of h if measured at sufficient depth in the system. In ref. [2,8] has been proposed that $P(v)$ is characterised, in a very

broad class of models, by the following behaviour, which is approximately recovered in the present model (we find deviations at very small v):

$$P(v) = Av^{\alpha} \exp(-v/\phi) \tag{9}$$

An interesting result from our numerical calculation, consistent with the theoretical considerations presented above, is that the asymptotic exponential behaviour of $P(v)$ is recovered for all values of $\delta > 0$. The correlations present in force transmission between grains, except exceptional cases, do not alter this property. The validity of this important observation may be broader, being consistent with results obtained in different contexts as the carbon paper experiment with compressed stationary bead packs of ref. [2] or experiments on continuously sheared granular materials of ref. [3], or in numerical simulations of scalar or vectorial force models [8,12,13,15].

However, the quantities α and ϕ, which in mean field theory seem to be exclusively related to the coordination number of the lattice of grain packing [8,12], strongly depend on the degree of correlation present in forces transmission, i.e., in our model, on the fraction δ. This fact has important practical consequences and may explain contrasting results found in different experiments. The dependence of the distribution $P(v)$ on spatial correlations has been recently experimentally outlined in ref. [3].

In the present model, the exponent α decreases in presence of spatial correlations in force transmission between grains. It passes from the value $\alpha = 1$ at $\delta = 1$ (as predicted by mean field theory) to the value $\alpha \simeq -1.1$ at $\delta = 0$ (in agreement with the simulations of ref. [9]). At a fraction of "normal" sites $\delta \sim 0.7$, a value higher than the percolation threshold on the square lattice, α crosses the zero. The exponent α is depicted in Fig. 5 as a function of the vertical correlation length ξ_V. Figure 5 shows that in the region of not too large values of ξ_V, small changes in ξ_V may induce large variations of α. Such a strong dependence seems to be found also in the experimental observations of ref. [3], where was shown the sensitivity of the measure of α with respect to the relative sizes of the grains and of the measuring device. In particular, from the above general discussion and the results of ref. [3], we expect that the results of mean field theory have to be experimentally recovered essentially when the measures of forces are taken averaging over regions which are larger than the length ξ_V.

For what concerns the parameter ϕ, Fig. 5 shows that it diverges approximately as a power law of ξ_V. The superimposed fit, consistent with the previous theoretical arguments, is $\phi(\xi_V) \sim \xi_V^{1.0}$. Thus ϕ grows with ξ_V and in the limit of huge spatial correlation, i.e., $\xi_V \to \infty$ (or $\delta \to 0$), the exponential asymptotic decay of eq. (9) is lost and just the power law behaviour survives. This fact has significant practical importance because it implies that giant stress fluctuations can be observed [9].

In conclusion, we have studied a simple microscopic model in the framework of scalar force approximation [8,9], which schematically takes into account the effects of correlations present in forces transmission in granular assemblies. Its simplicity allows to clarify the effects induced by the extension of "forces chains" in the system. The presence of such a non trivial characteristic length scale, which is experimentally measured for instance by photoelastic visualization, can thus be quantitatively related to other measurable quantities as the forces distributions, $P(v)$. We have discussed some interesting correspondences with known results from recent experiments [3]. However further experimental and theoretical investigation about these important effects is still missing.

Interestingly the present model is related (see ref. [8]) to works devoted to describe other physical phenomena as directed Abelian sandpiles, aggregation-dissociation reactions, interface dynamics, or river networks [16].

Acknowledgements It's a pleasure to thank Prof. R. P. Behringer for very fruitful discussions and his critical reading of the manuscript.

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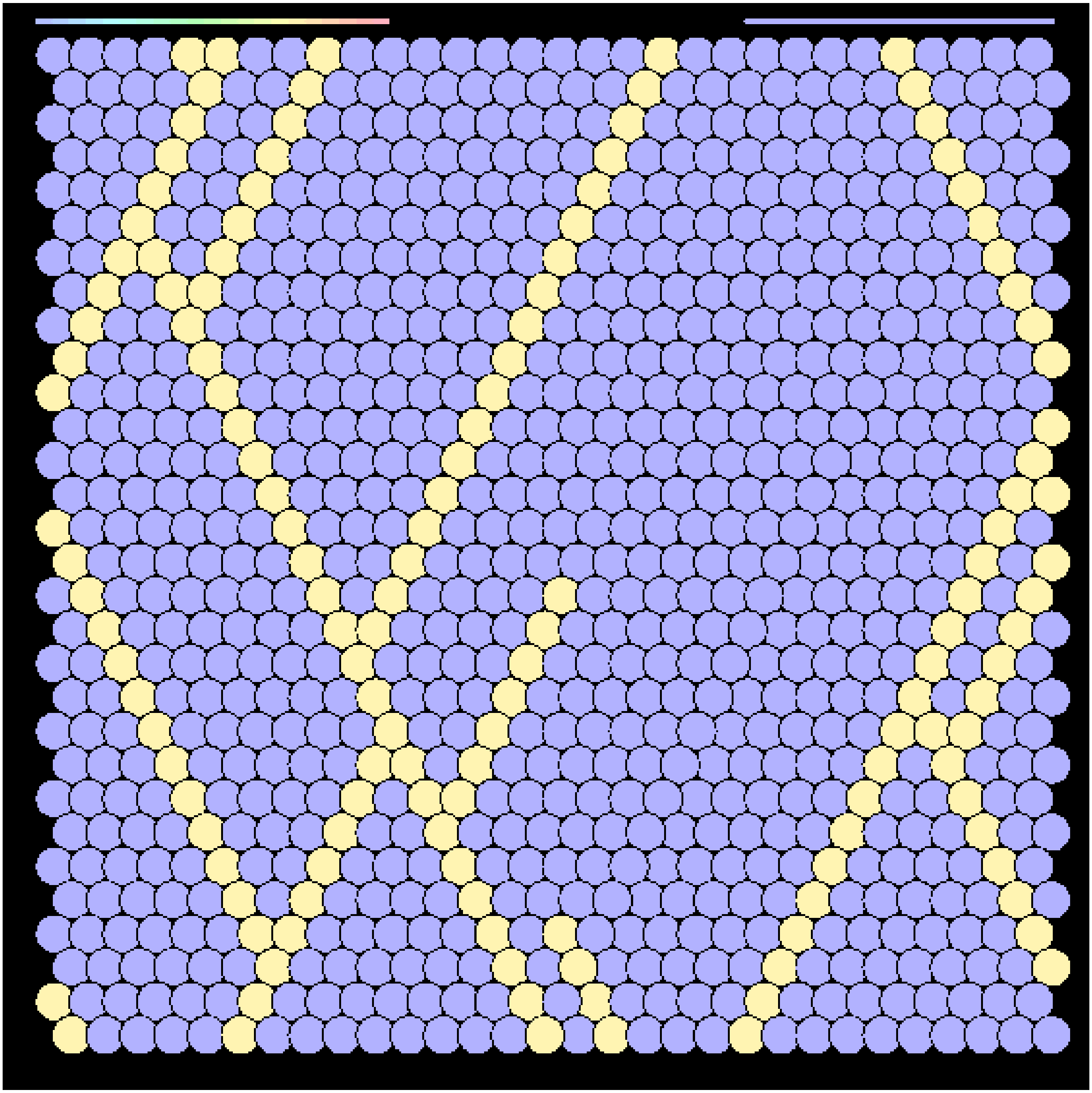
FIG. 1. Two pictures of the conformation of stress paths in our model, corresponding to two values of the correlation length, ξ_V ($\xi_V \sim 2$ in the left picture, and $\xi_V \sim 20$ on the right). They are obtained enlightening the grains which carry a weight above a given threshold.

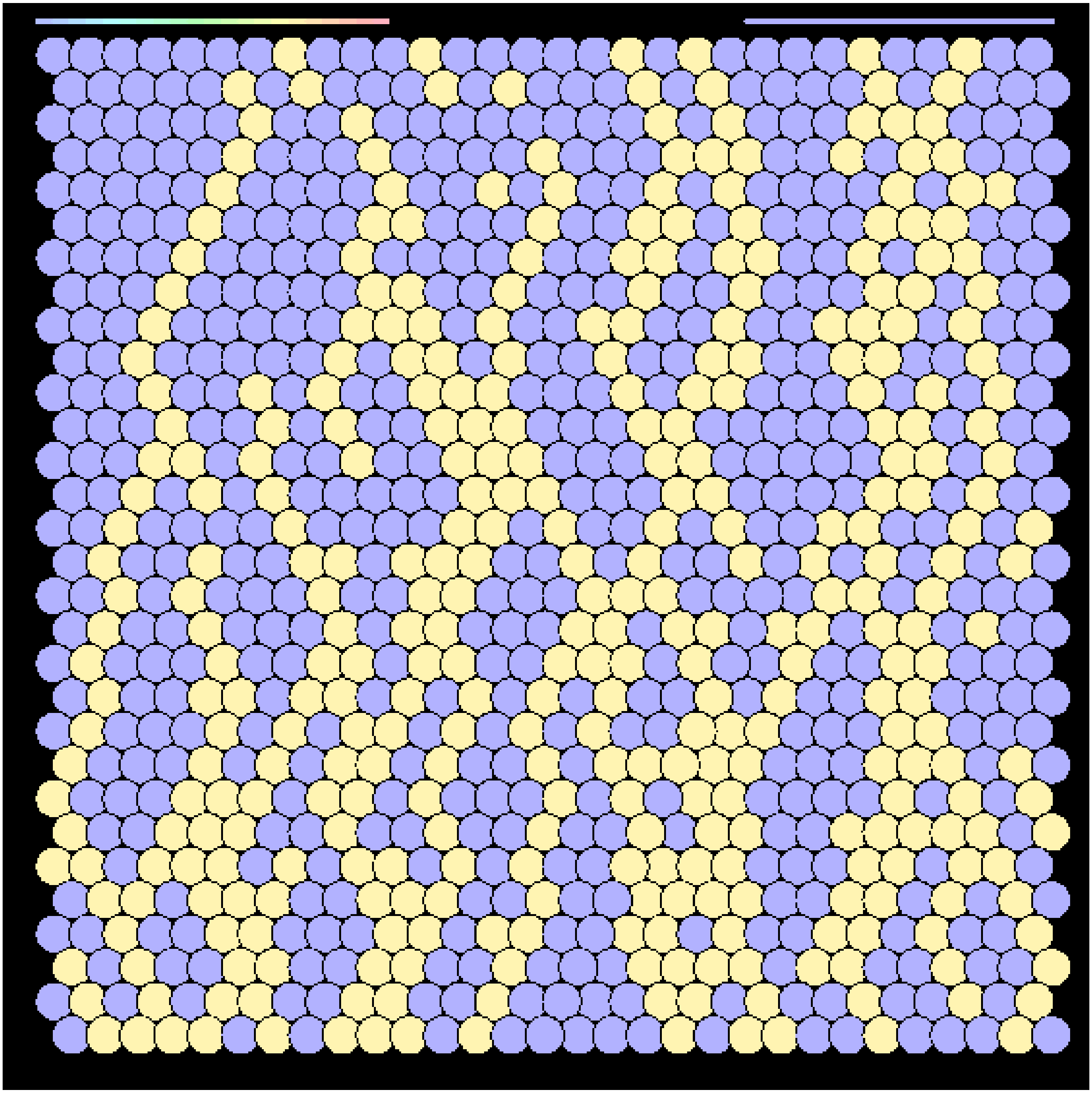
FIG. 2. The vertical, $C_V(r)$, and the horizontal, $C_H(r)$, force correlation functions as a function of the grain distance r (in unit of the lattice spacing), for several values of the fraction of grains not undergoing the "slip condition" of eq. (2), $\delta = 0.1, 0.3, 0.6, 1.0$ (resp. squares, triangles, diamonds, circles). The superimposed curves are the exponential fits described in the texts.

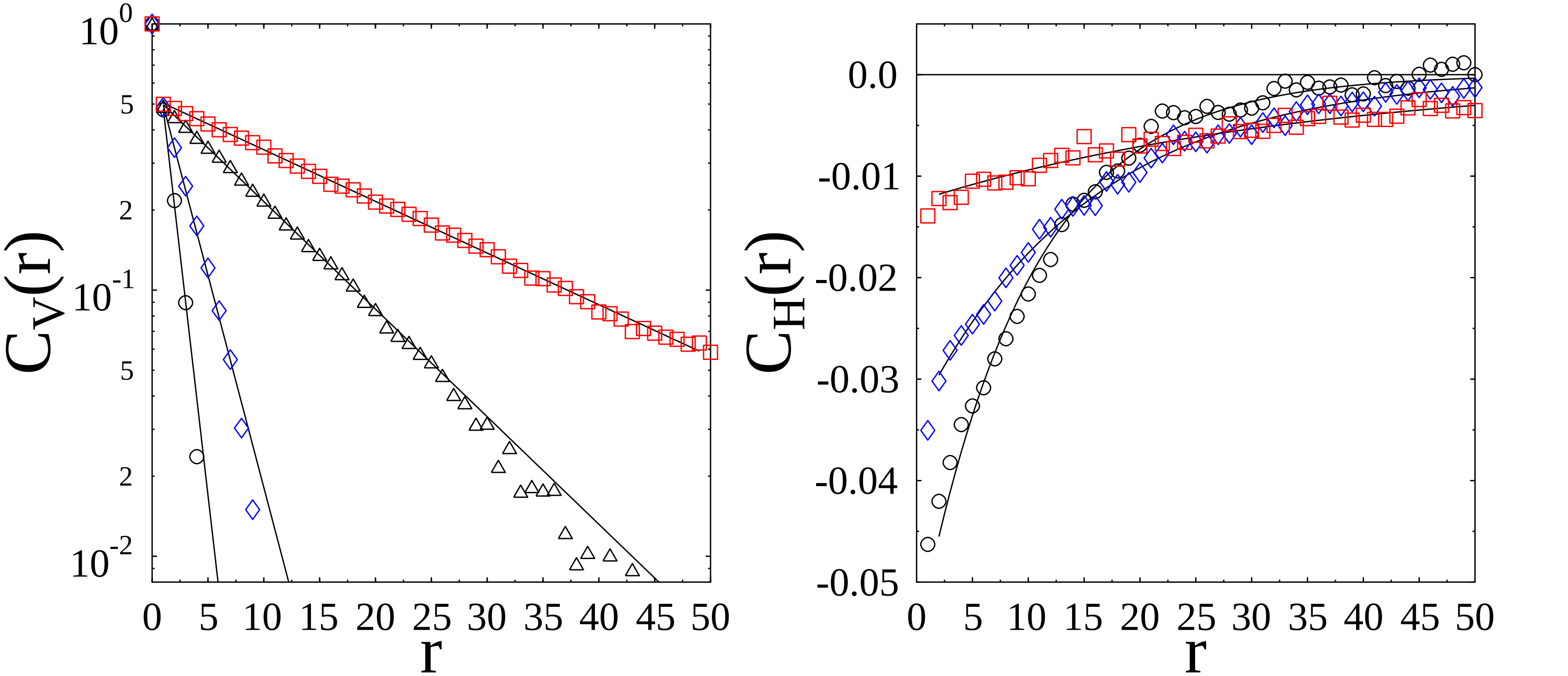
FIG. 3. *Left:* the characteristic length ξ_V of the vertical force correlation function as a function of δ. It diverges when $\delta \to 0$ approximately as $\xi_V \sim \delta^{-1.0}$. In such a limit force correlations extend over the whole system, and this fact strongly affects forces distribution. *Right:* the characteristic length ξ_H of the horizontal force correlation function as a function of ξ_V ($\xi_H \sim \xi_V^{1/2}$).

FIG. 4. The distribution $P(v)$ of forces $v = w/w_m$ normalized by the mean force w_m at the bottom of our system, depicted for three values of the fraction of grains not undergoing the "slip condition" of eq. (2), $\delta = 0.1, 0.6, 1.0$ (resp. squares, diamonds, circles). The superimposed fits are from eq. (9).

FIG. 5. The parameters of eq. (9), α and ϕ, as a function of the correlation length ξ_V. *Right:* the exponent α passes from the value predicted by mean field theory, $\alpha = 1$, at small ξ_V (i.e., $\delta = 1$) to $\alpha \simeq -1.1$ when $\xi_V \to \infty$ (i.e., at $\delta = 0$). The sensitivity of α to changes of ξ_V remembers the observations from experiments by Miller et al. [3]. *Left:* the parameter ϕ diverges as a power law with ξ_V (approx. $\phi \sim \xi_V$), showing that if $\xi_V \to \infty$ the exponential asymptotic decay of force distribution $P(v)$ is lost, and huge stress fluctuations are possible.







C_V(r)
r
C_H(r)
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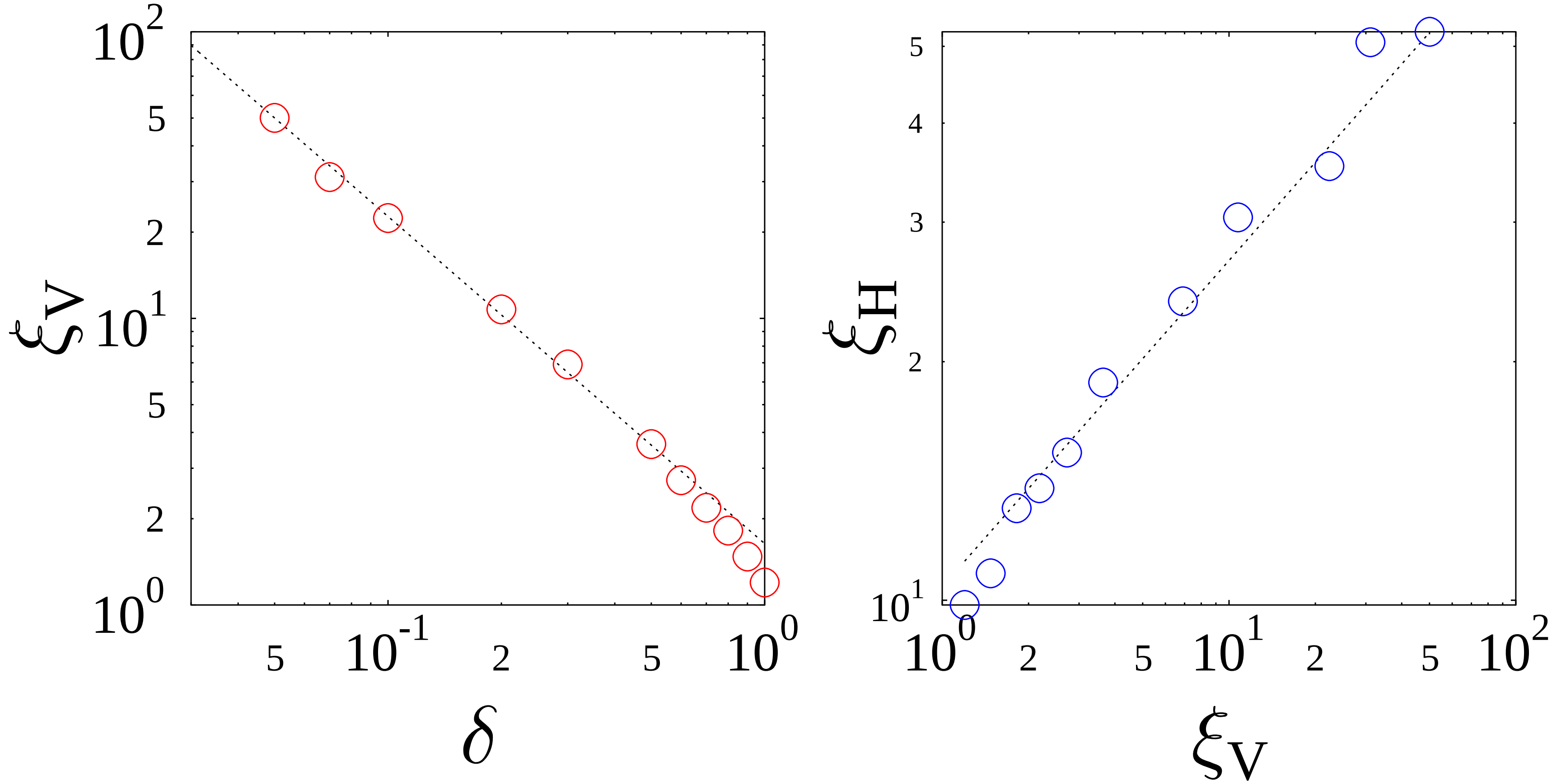
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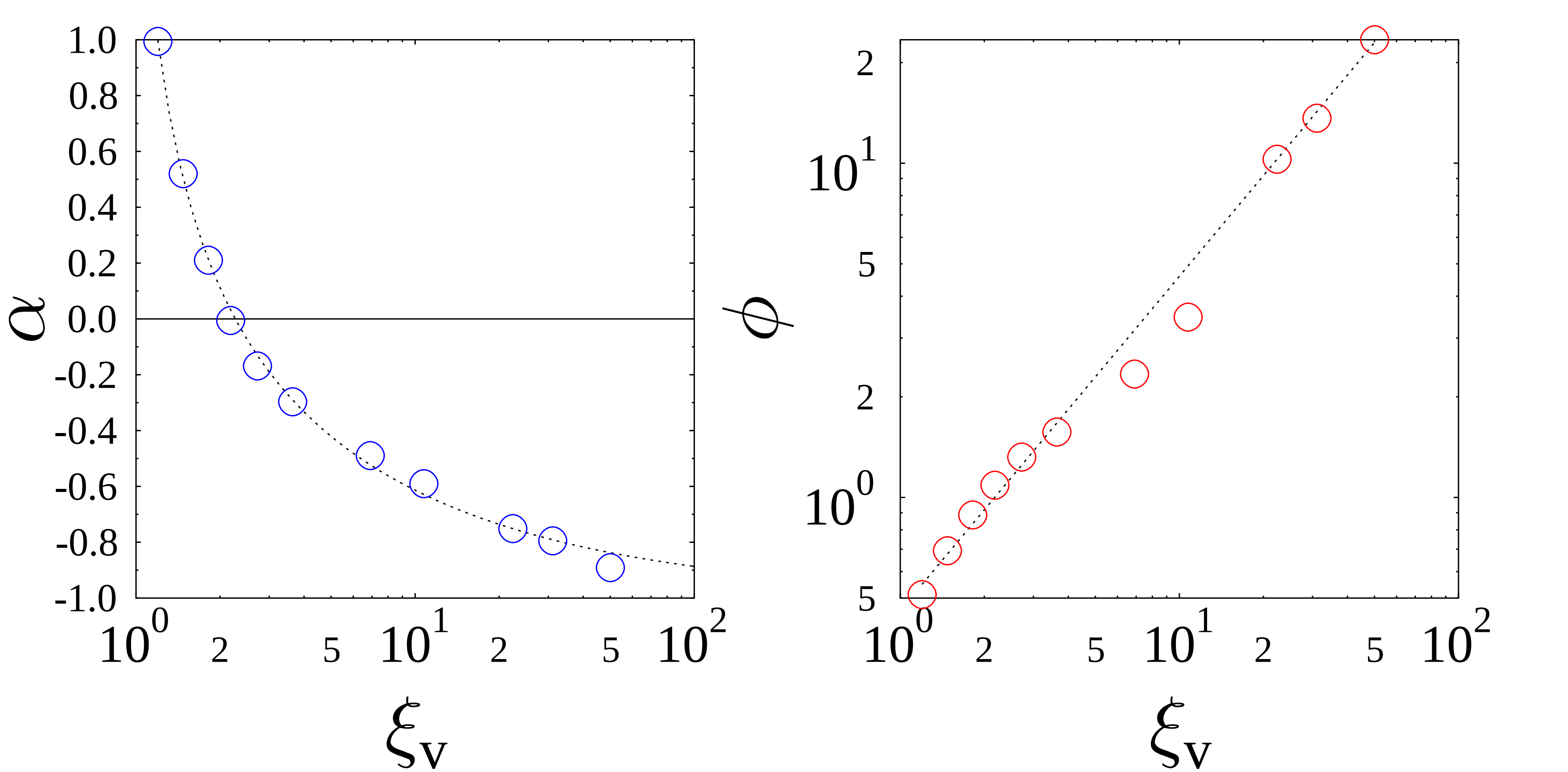

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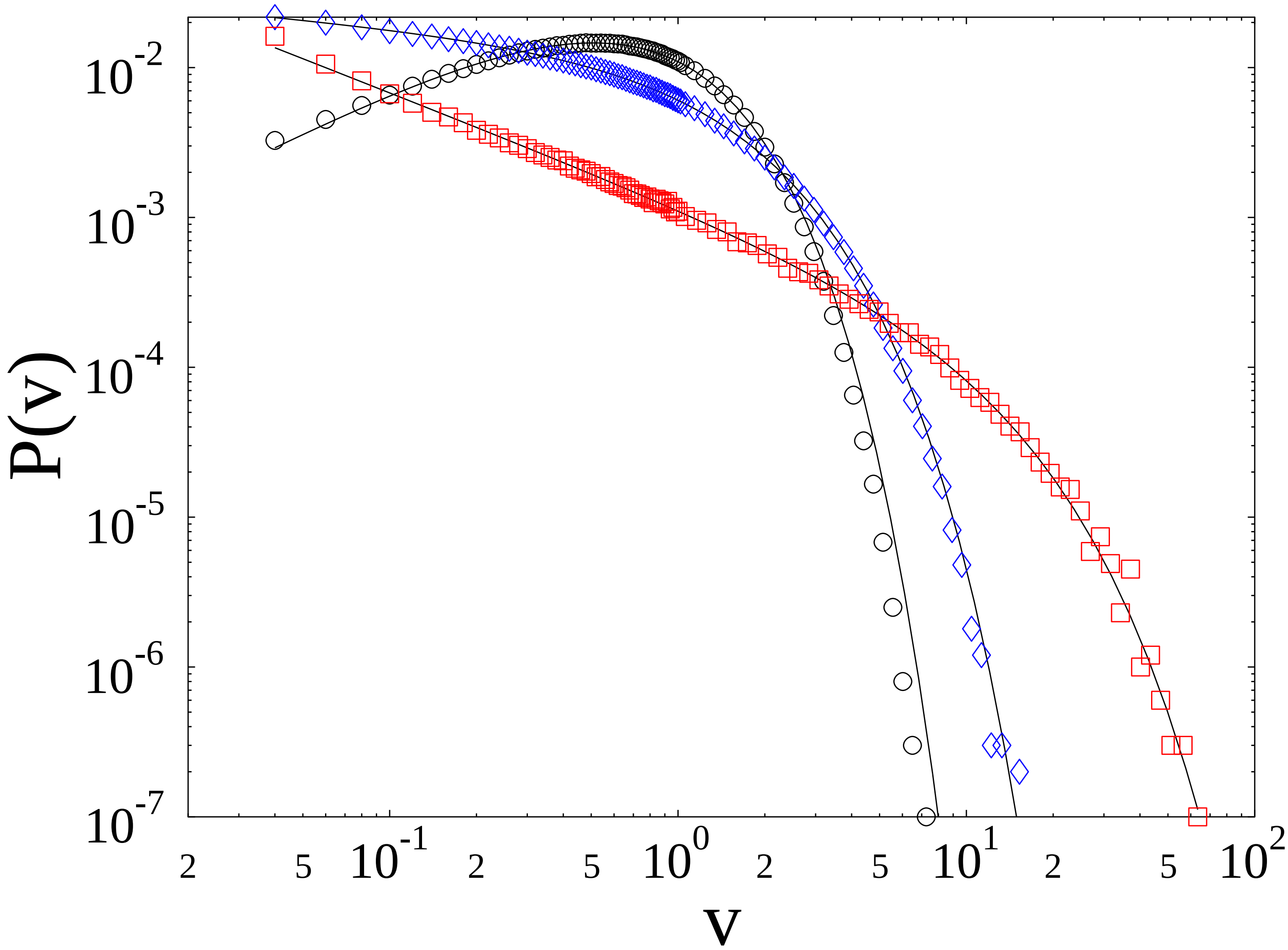
P(v)
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